

Integration by Parts Formula, Derivative Formula, and Transportation Inequalities for SDEs Driven by Fractional Brownian Motion

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Abstract. This paper is devoted to study a stochastic differential equation driven by a fractional Brownian motion in the framework of [L. Coutin and L. Decreusefond, Stochastic differential equations driven by a fractional Brownian motion, Tech. Report 97C004, ENST Paris, 1997]. We first prove the Driver integration by parts formula and as a consequence, provide an alternative proof for a regular result of L. Coutin and L. Decreusefond. Secondly, by using the techniques of Malliavin calculus, the Bismut derivative formula is established. Furthermore, the formula is applied to the study of estimate gradient and strong Feller property. Finally, we show that the Talagrand type transportation cost inequalities hold on the path space with respect to both the uniform metric and the L^2 -metric.

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1 Introduction

The Driver integration by parts formula [12] and the Bismut derivative formula [4] are two quite useful tools in various aspects of stochastic analysis. Let ∇ be the gradient operator and P_t stand for the diffusion semigroup. The above two formulas allow us to estimate the commutator $\nabla P_t - P_t \nabla$, which plays a key role in the study of flow properties [16]. On the other hand, [39] showed that, in general the integration by parts formula is more complicated and harder to obtain than the derivative formula. Based on martingale method, coupling argument or Malliavin calculus, the derivative formula has been widely studied and applied in various fields, such as heat kernel estimates, strong Feller property and functional inequalities, see [13, 37, 41, 43] and references therein. Whereas, in [39], based upon a new coupling argument, the integration by parts formulae are derived and applied to various models including degenerate diffusion process, delayed SDEs and semi-linear SPDEs.

Recently, transportation cost inequality has been widely studied. Let (E, d) be a metric space equipped with σ -algebra \mathcal{B} such that $d(\cdot, \cdot)$ is $\mathcal{B} \times \mathcal{B}$ measurable. For any $p \geq 1$ and

two probability measures μ and ν on (E, \mathcal{B}) , the L^p -Wasserstein distance induced by d between these two probability measures is defined by

$$W_p^d(\mu, \nu) = \inf_{\pi \in \mathcal{C}(\mu, \nu)} \left(\int_E \int_E d(x, y)^p \pi(dx, dy) \right)^{1/p},$$

where $\mathcal{C}(\mu, \nu)$ denotes the set of all coupling of μ and ν . In 1996, Talagrand [31] proved the following transportation cost inequality for the standard Gaussian measure μ on \mathbb{R}^d :

$$W_2^d(f\mu, \mu)^2 \leq 2\mu(f \log f), \quad f > 0, \mu(f) = 1,$$

where $d(x, y) = |x - y|$. In general, we call that the probability measure μ satisfies the L^p -transportation cost inequality on (E, d) , if there exists a constant $C(\geq 0)$ such that for any probability measure ν ,

$$W_p^d(\mu, \nu) \leq \sqrt{2C\mathbb{H}(\nu|\mu)}, \quad (1.1)$$

where $\mathbb{H}(\nu|\mu)$, the relative entropy of ν with respect to μ , is given by

$$\mathbb{H}(\nu|\mu) = \begin{cases} \int_E \frac{d\nu}{d\mu} \log \frac{d\nu}{d\mu} d\mu, & \text{if } \nu \ll \mu, \\ +\infty, & \text{else.} \end{cases}$$

For simplicity, we write $\mu \in T_p(C|d)$ for (1.1). In the past decades, the work of Talagrand has been generalized to various different stochastic processes, see, for instance, [27, 5] for the Hamilton-Jacobi equation, [17, 18] on abstract Wiener space, [15] on loop groups, [11, 34, 35, 36] for diffusion processes, [33] for SDEs of pure jumps, [21] for SDEs driven by both Brownian motion and jump process, [3] for neutral functional SDEs, [30] for SDEs driven by fractional Brownian motion.

In this article, we are interested in the stochastic differential equations driven by a fractional Brownian motion. It is well known that the main difficult point raised by the fractional Brownian motion is that it is not Markovian process and semimartingale, so the Itô approach to setup a stochastic integral with respect to the fractional Brownian motion is not valid. Now there exist numerous attempts to define a stochastic integral with respect to the fractional Brownian motion and moreover, many works to discuss the stochastic differential equations driven by a fractional Brownian motion. We briefly present some results. Based on a fractional integration by parts formula [42], Nualart and Răşcanu [25] established the existence and uniqueness result with $H > \frac{1}{2}$. By using the theory of rough path analysis introduced in [20], Coutin and Qian [7] proved an existence and uniqueness result with Hurst parameter $H \in (\frac{1}{4}, \frac{1}{2})$. For the regularity results about the law of the solution, one can see [19, 23, 26] and references therein.

The equation driven by a fractional Brownian motion we are to deal with is of Volterra type, which is originally discussed by Coutin and Decreusefond [6]. In [6], they studied existence, uniqueness and regularity of solution. In this paper, by using coupling argument and the Girsanov transform for fractional Brownian motion, we first prove the Driver integration by parts formula for aimed equation. As an important application, we give an alternative proof for [6, Corollary 4.1]. Secondly, the Bismut derivative formula is established, using the techniques of Malliavin calculus. As applications, gradient estimate, the Harnack inequality and the strong Feller property are derived. Finally, we obtain the Talagrand type transportation cost inequalities for the law of the solution of aimed equation on the path space with respect to both the uniform metric and L^2 -metric.

The paper is organized as follows. In section 2, we give some preliminaries on fractional Brownian motion. In section 3, we investigate the Driver integration by parts formula, while in section 4, the Bismut derivative formula is discussed. Finally, section 5 is devoted to the transportation cost inequalities.

2 Preliminaries

Let $B^H = \{B_t^H, t \in [0, T]\}$ be a fractional Brownian motion with Hurst parameter $H \in (0, 1)$ defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Namely, B^H is a centered Gauss process with the covariance function

$$\mathbb{E}(B_t^H B_s^H) = R_H(t, s) := \frac{1}{2} (t^{2H} + s^{2H} - |t - s|^{2H}).$$

When $H = \frac{1}{2}$, the process $B^{\frac{1}{2}}$ is the usual Brownian motion. By the above covariance function and the Kolmogorov criterion, we know that B^H have $(H - \epsilon)$ -order Hölder continuous paths for all $\epsilon > 0$. Furthermore, B^H has stationary increments and is self-similar with Hurst index H .

From [10], it is known that the covariance kernel $R_H(t, s)$ admits the following representation:

$$R_H(t, s) = \int_0^{t \wedge s} K_H(t, r) K_H(s, r) dr,$$

where $K_H(\cdot, \cdot)$ is a square integrable kernel given by

$$K_H(t, s) = \Gamma\left(H + \frac{1}{2}\right)^{-1} (t - s)^{H - \frac{1}{2}} F\left(H - \frac{1}{2}, \frac{1}{2} - H, H + \frac{1}{2}, 1 - \frac{t}{s}\right),$$

in which $F(\cdot, \cdot, \cdot, \cdot)$ is the Gauss hypergeometric function (for details, see [22]).

Again by [10], the operator $K_H : L^2([0, T]; \mathbb{R}) \rightarrow I_{0+}^{H+1/2}(L^2([0, T]; \mathbb{R}))$ associated with the kernel $K_H(\cdot, \cdot)$ is defined as follows

$$(K_H f)(t) := \int_0^t K_H(t, s) f(s) ds,$$

where $I_{0+}^{H+1/2}$ is the $(H + 1/2)$ -order left fractional Riemann-Liouville integral operator on $[0, T]$. It is an isomorphism and for each $f \in L^2([0, T]; \mathbb{R})$,

$$(K_H f)(s) = I_{0+}^{2H} s^{1/2-H} I_{0+}^{1/2-H} s^{H-\frac{1}{2}} f, \quad H \leq 1/2, \quad (2.1)$$

$$(K_H f)(s) = I_{0+}^1 s^{H-1/2} I_{0+}^{H-1/2} s^{1/2-H} f, \quad H \geq 1/2. \quad (2.2)$$

Hence, for any $h \in I_{0+}^{H+1/2}(L^2([0, T]; \mathbb{R}))$, the inverse operator K_H^{-1} can be written as

$$\begin{aligned} (K_H^{-1} h)(s) &= s^{H-1/2} D_{0+}^{H-1/2} s^{1/2-H} h', \quad H > 1/2, \\ (K_H^{-1} h)(s) &= s^{1/2-H} D_{0+}^{1/2-H} s^{H-1/2} D_{0+}^{2H} h, \quad H < 1/2, \end{aligned}$$

where D_{0+}^α is the α -order left-sided Riemann-Liouville derivative operator, $\alpha \in (0, 1)$.

In particular, when h is absolutely continuous, it holds

$$(K_H^{-1} h)(s) = s^{H-1/2} I_{0+}^{1/2-H} s^{1/2-H} h', \quad H < 1/2.$$

For more details about the deterministic fractional calculus, one can refer to [29].

We assume that Ω is the canonical probability space $C_0([0, T]; \mathbb{R})$, the set of continuous functions, null at time 0, equipped with the Borel σ -algebra and \mathbb{P} is the law of the fractional Brownian motion. The canonical filtration is $\mathcal{F}_t = \sigma\{B_s^H : 0 \leq s \leq t\} \vee \mathcal{N}$, where \mathcal{N} is the set of the \mathbb{P} -null sets. According to [10, Theorem 3.3], the Cameron-Martin space of the fractional Brownian motion, denoted by \mathcal{H} , is equal to $I_{0+}^{H+1/2}(L^2([0, T]; \mathbb{R}))$, i.e., for any $h \in \mathcal{H}$, it can be represented as $h(t) = K_H \dot{h}(t)$, where the function \dot{h} belongs to $L^2([0, T]; \mathbb{R})$. The scalar product on \mathcal{H} is defined by

$$(h, g)_{\mathcal{H}} := (K_H^{-1}h, K_H^{-1}g)_{L^2([0, T]; \mathbb{R})}, \quad \forall h, g \in \mathcal{H}.$$

As a consequence, $(\Omega, \mathcal{H}, \mathbb{P})$ is an abstract Wiener space in the sense of Gross. Furthermore, let Ω^* denote the strong topological dual of Ω , then there hold

$$\Omega^* \xrightarrow{K_H^*} L^2([0, T]; \mathbb{R}) \xrightarrow{K_H} \mathcal{H} \xrightarrow{i_H} \Omega$$

and

$$R_H = K_H \circ K_H^*,$$

where we identify the operator R_H and its kernel.

Next we summarize some basic results of Malliavin calculus associated with the fractional Brownian motion, and we refer to [10], [24] and [32] for a comprehensive presentation.

Let \mathcal{S} denote the set of smooth and cylindrical random variables of the form:

$$F = f(\langle l_1, \omega \rangle, \dots, \langle l_n, \omega \rangle)$$

where $n \geq 1$, $f \in C_b^\infty(\mathbb{R}^n)$, the set of f and all its partial derivatives are bounded, $l_i \in \Omega^*$, $1 \leq i \leq n$. The Malliavin derivative of F , denoted by $D_H F$, is defined as the \mathcal{H} -valued random variable

$$D_H F(\omega) = \sum_{i=1}^n \frac{\partial f}{\partial x_i}(\langle l_1, \omega \rangle, \dots, \langle l_n, \omega \rangle) R_H(l_i).$$

For any $k \in \mathbb{N}$, denote by D_H^k the iteration of D_H . For any $p \geq 1$ and $k \in \mathbb{N}$, we define the Sobolev space $\mathbb{D}_H^{k,p}$ as the completion of \mathcal{S} with respect to the norm:

$$\|F\|_{k,p}^p := \mathbb{E}|F|^p + \mathbb{E} \sum_{i=1}^k \|D_H^i F\|_{\mathcal{H}^{\otimes i}}^p.$$

The divergence operator δ_H , also called the Skorohod integral, is defined by using the duality relationship. More precisely, the domain $\text{Dom}_p \delta_H$ is the set of process u such that

$$|\mathbb{E}\langle D_H F, u \rangle_{\mathcal{H}}| \leq C(\mathbb{E}|F|^q)^{\frac{1}{q}}$$

for all $F \in \mathbb{D}_H^{1,q}$, where q satisfies $1/p + 1/q = 1$ and C is some constant depending on u . If $u \in \text{Dom}_p \delta_H$, then $\delta_H u$ is defined by

$$\mathbb{E}(F \delta_H u) = \mathbb{E}\langle D_H F, u \rangle_{\mathcal{H}}.$$

It is well known that, in the case of the Brownian motion ($H = 1/2$), the Skorohod integral is an extension of the Itô integral. So, this motivates us to use the divergence operator to define a stochastic integral with respect to the fractional Brownian motion. That is,

$$\int_0^T u_t \delta_H B_t^H := \delta_H(K_H u),$$

where the process $K_H u \in \text{Dom} \delta_H := \cup_{p \geq 1} \text{Dom}_p \delta_H$ (see e.g. [10] and [6]). According to [10, Theorem 4.8], we have the following Lévy-Hida representation:

$$W := \left(\int_0^t \mathbf{I}_{[0,t]}(s) \delta_H B_s^H \right)_{0 \leq t \leq T} = (\delta_H(K_H \mathbf{I}_{[0,t]}))_{0 \leq t \leq T}$$

is a standard Brownian motion whose filtration is equal to $\{\mathcal{F}_t, 0 \leq t \leq T\}$ and moreover, for any $u \in L_a^2([0, T] \times \Omega)$, the set of square integrable and adapted processes, it holds

$$\int_0^t u_s \delta_H B_s^H = \int_0^t u_s dW_s, \forall t \in [0, T].$$

In particular, for each $t \in [0, T]$, taking $u = K_H(t, \cdot)$, then we get $B_t^H = \int_0^t K_H(t, s) dW_s$.

In the paper, we are concerned with a \mathbb{R} -valued equation driven by a fractional Brownian motion of the form:

$$X_t = x + \int_0^t K_H(t, s) b(s, X_s) ds + \int_0^t K_H(t, s) \sigma(s, X_s) dW_s. \quad (2.3)$$

Note that, when $\sigma \equiv C$, then the third term of the right-hand side of (2.3) is equal to $C B_t^H$. Hence, the factor $K_H(t, s)$ in the noise term is necessary to make the equation sense. While $K_H(t, s)$ in the drift term is only to symmetrize b and σ . Set $H_0 = |H - 1/2|$, $A_H = \{p \geq 1 : pH_0 < 1\}$ and for every $p \in A_H$, put $\kappa_p = (1 - pH_0)^{-1}$, $L_+^{\kappa_p} = \cup_{q > \kappa_p} L^q([0, T]; \mathbb{R})$.

Definition 2.1 A \mathbb{R} -valued process $(X_t)_{t \in [0, T]}$ is called a solution of (2.3), if it is adapted such that $\mathbb{E}|X_t|^2 \in L_+^{\kappa_2}$, and (2.3) is satisfied $d\mathbb{P} \times dt$ a.s.

Remark 2.2 From [6, Theorem 3.1 and Theorem 3.2], we know that, if b and σ are Lipschitz continuous for the second variable uniformly with respect to their first variable, and there exist $x_0, y_0 \in \mathbb{R}$ such that $b(\cdot, x_0) \in L_+^{\kappa_1}([0, T]; \mathbb{R})$, $\sigma(\cdot, y_0) \in L_+^{2\kappa_2}([0, T]; \mathbb{R})$, then (2.3) has a unique solution. Furthermore, for all $p \in A_H$, $\sup_{t \in [0, T]} \mathbb{E}|X_t|^p < \infty$. If σ is bounded, then X has almost surely continuous trajectories.

Define $P_t f(x) := \mathbb{E}f(X_t^x)$, $t \in [0, T]$, $f \in \mathcal{B}_b(\mathbb{R})$, where X_t^x is the solution to (2.3) with $X_0 = x$ and $\mathcal{B}_b(\mathbb{R})$ denotes the set of all bounded measurable functions on \mathbb{R} . Besides, we denote by $C_b^1(\mathbb{R})$ the set of all bounded continuous differentiable functions. In the remainder of the paper, we will establish the integration by parts formula and the Bismut derivative formula for P_T , and moreover obtain the Talagrand type transportation cost inequalities for the law of the solution of (2.3) on the path space.

3 Integration by parts formula

This section is devoted to the equation (2.3) with additive noise, i.e.

$$X_t = x + \int_0^t K_H(t, s)b(s, X_s)ds + \int_0^t K_H(t, s)\sigma(s)dW_s. \quad (3.1)$$

We aim to establish integration by parts formula by the method of coupling and Girsanov transformation. As an application, we give an alternative proof for [6, Corollary 4.1], in which the absolute continuity of the law of the solution is discussed.

To start with, let us give some conditions of the coefficients b and σ : (H1)

- (i) b is continuously differentiable w.r.t. the second variable and there exist positive constants K_1 and K_2 such that

$$|\partial b(t, \cdot)(x)| \leq K_1, |\sigma(t)^{-1}| \leq K_2, \quad \forall t \in [0, T], x \in \mathbb{R};$$

- (ii) there exist $x_0 \in \mathbb{R}$ such that $b(\cdot, x_0) \in L_+^{\kappa_1}([0, T]; \mathbb{R}), \sigma(\cdot) \in L_+^{2\kappa_2}([0, T]; \mathbb{R})$.

Theorem 3.1 *Let $T > 0$ and $y \in \mathbb{R}$ be fixed. Assume that (H1) holds.*

- (1) *For each $f \in C_b^1(\mathbb{R})$, there holds the integration by parts formula*

$$P_T(\nabla_y f)(x) = \mathbb{E} \left[f(X_T^x) \int_0^T \sigma(s)^{-1} \left(C_H s^{\frac{1}{2}-H} - s \partial b(s, \cdot)(X_s) \right) \frac{y}{T} dW_s \right],$$

where C_H is a positive constant given in the proof below.

As a consequence, for each $\alpha > 0$ and positive $f \in C_b^1(\mathbb{R})$,

$$\begin{aligned} |P_T(\nabla_y f)| &\leq \alpha [P_T(f \log f) - (P_T f)(\log P_T f)] \\ &\quad + \frac{K_2^2 y^2}{\alpha} \left(\frac{C_H^2}{(2-2H)T^{2H}} + \frac{4C_H K_1}{5-2H} T^{\frac{1}{2}-H} + \frac{K_1^2 T}{3} \right) P_T f. \end{aligned}$$

- (2) *For each non-negative $f \in \mathcal{B}_b(\mathbb{R})$, there holds the shift Harnack inequality*

$$(P_T f)^p \leq (P_T \{f(y + \cdot)\})^p \exp \left[\frac{pK_2^2}{p-1} \left(\frac{C_H^2}{(2-2H)T^{2H}} + \frac{4C_H K_1}{5-2H} T^{\frac{1}{2}-H} + \frac{K_1^2 T}{3} \right) y^2 \right].$$

- (3) *For each positive $f \in \mathcal{B}_b(\mathbb{R})$, there holds the shift log-Harnack inequality*

$$P_T \log f \leq \log P_T \{f(y + \cdot)\} + K_2^2 \left(\frac{C_H^2}{(2-2H)T^{2H}} + \frac{4C_H K_1}{5-2H} T^{\frac{1}{2}-H} + \frac{K_1^2 T}{3} \right) y^2.$$

Proof. Obviously, by (H1), it follows from Remark 2.2 that (3.1) has a unique solution. On the other hand, for any $\epsilon \in [0, 1]$, let X_t^ϵ solve the equation

$$X_t^\epsilon = x + \int_0^t K_H(t, s)b(s, X_s)ds + \int_0^t K_H(t, s)\sigma(s)dW_s + \frac{t\epsilon}{T}y, \quad t \in [0, T]. \quad (3.2)$$

It is easy to see that $X_t^\epsilon = X_t + \frac{t\epsilon}{T}y$, $t \in [0, T]$. In particular, $X_T^\epsilon = X_T + \epsilon y$.
Next, let

$$C_H = \begin{cases} \frac{\Gamma(2H)\Gamma(1/2-H)}{B(2-2H, 2H)} \left(\frac{1}{2} - H\right), & H < \frac{1}{2}; \\ 1, & H = \frac{1}{2}; \\ \frac{\Gamma(H-1/2)}{B(2-2H, H-1/2)}, & H > \frac{1}{2}. \end{cases}$$

It follows from (2.1) and (2.2) that, for each $H \in (0, 1)$, $K_H(C_H x^{1/2-H})(t) = t$. Therefore, we can reformulate (3.2) as

$$X_t^\epsilon = x + \int_0^t K_H(t, s)b(s, X_s^\epsilon)ds + \int_0^t K_H(t, s)\sigma(s)dW_s^\epsilon, \quad t \in [0, T],$$

where $W_s^\epsilon = W_s + \int_0^s \sigma(r)^{-1} \left(b(r, X_r) - b(r, X_r^\epsilon) + \frac{C_H \epsilon y}{T} r^{\frac{1}{2}-H} \right) dr$, $s \in [0, T]$.

Let

$$\xi_\epsilon(r) = b(r, X_r) - b(r, X_r^\epsilon) + \frac{C_H \epsilon y}{T} r^{\frac{1}{2}-H}$$

and

$$R_\epsilon = \exp \left[- \int_0^T \sigma(r)^{-1} \xi_\epsilon(r) dW_r - \frac{1}{2} \int_0^T |\sigma(r)^{-1} \xi_\epsilon(r)|^2 dr \right].$$

According to (H1), we easily get $\mathbb{E} \exp \left[\frac{1}{2} \int_0^T |\sigma(r)^{-1} \xi_\epsilon(r)|^2 dr \right] < \infty$. By the Novikov condition and the Girsanov theorem, $(W_t^\epsilon)_{0 \leq t \leq T}$ is a Brownian motion under the probability measure $\mathbb{Q}_\epsilon := R_\epsilon \mathbb{P}$. Then (X, X^ϵ) is a coupling by change of measure with changed probability \mathbb{Q}_ϵ . Since $R_0 = 1$, by [39, Theorem 2.1], to obtain the desired integration by parts formula, it remains to confirm the following equality: in the sense of $L^1(\mathbb{P})$,

$$\frac{d}{d\epsilon} R_\epsilon|_{\epsilon=0} = - \int_0^T \sigma(r)^{-1} \left[C_H r^{\frac{1}{2}-H} - r \partial b(r, \cdot)(X_r) \right] \frac{y}{T} dW_r.$$

Actually, noting that

$$\begin{aligned} \lim_{\epsilon \rightarrow 0} \mathbb{E} \frac{R_\epsilon - 1}{\epsilon} &= \lim_{\epsilon \rightarrow 0} \mathbb{E} \frac{- \int_0^T \sigma(r)^{-1} \xi_\epsilon(r) dW_r - \frac{1}{2} \int_0^T |\sigma(r)^{-1} \xi_\epsilon(r)|^2 dr}{\epsilon} \\ &= \lim_{\epsilon \rightarrow 0} \mathbb{E} \frac{- \int_0^T \sigma(r)^{-1} \xi_\epsilon(r) dW_r}{\epsilon}, \end{aligned}$$

and, moreover

$$\begin{aligned} &\mathbb{E} \left| \frac{- \int_0^T \sigma(r)^{-1} \xi_\epsilon(r) dW_r}{\epsilon} + \int_0^T \sigma(r)^{-1} \left[C_H r^{\frac{1}{2}-H} - r \partial b(r, \cdot)(X_r) \right] \frac{y}{T} dW_r \right| \\ &\leq K_2 \left[\mathbb{E} \int_0^T \left| \frac{b(r, X_r^\epsilon) - b(r, X_r) - \partial b(r, \cdot)(X_r) \frac{ry}{T} \epsilon}{\epsilon} \right|^2 dr \right]^{\frac{1}{2}}, \end{aligned}$$

so the dominated convergence theorem implies the assertion.

The second result in (1) follows by the given upper bounds on $|\sigma(t)^{-1}|$ and $|\partial b(t, \cdot)|$ and using the above integration by parts formula and the Young inequality (see, for instance, [2, Lemma 2.4])

$$|P_T(\nabla_y f)| - \alpha [P_T(f \log f) - (P_T f)(\log P_T f)]$$

$$\begin{aligned}
&\leq \alpha \log \mathbb{E} \exp \left[\frac{1}{\alpha} \int_0^T \sigma(s)^{-1} \left(C_H s^{\frac{1}{2}-H} - s \partial b(s, \cdot)(X_s) \right) \frac{y}{T} dW_s \right] \cdot P_T f \\
&\leq \frac{\alpha}{2} \log \mathbb{E} \exp \left[\frac{2}{\alpha^2} \int_0^T \left| \sigma(s)^{-1} \left(C_H s^{\frac{1}{2}-H} - s \partial b(s, \cdot)(X_s) \right) \frac{y}{T} \right|^2 ds \right] \cdot P_T f.
\end{aligned}$$

Finally, (2) and (3) can be easily derived by applying [39, Proposition 2.3] and the second inequality in (1). The proof is complete.

The shift Harnack inequality allows us to deduce the regularity for the law of the solution of (3.1). That is, we have the following result.

Corollary 3.2 *Suppose that the assumption (H1) holds. Then, for any $t > 0$, the law of X_t is absolutely continuous with respect to the Lebesgue measure.*

Proof. Without loss of generality, we only consider the case $t = T$. Let

$$C_1 = \frac{pK_2^2}{p-1} \left(\frac{C_H^2}{(2-2H)T^{2H}} + \frac{4C_H K_1}{5-2H} T^{\frac{1}{2}-H} + \frac{K_1^2 T}{3} \right).$$

The shift Harnack inequality stated in Theorem 3.1 implies, for any non-negative $f \in \mathcal{B}_b(\mathbb{R})$,

$$(P_T f(x))^p e^{-C_1 |y|^2} \leq (P_T \{f(y + \cdot)\}^p)(x).$$

Let A be a Lebesgue-null set, by applying the above inequality to $f = \mathbf{I}_A$ and noting the invariance property under shift for the Lebesgue measure, we have

$$(P_T \mathbf{I}_A(x))^p \int_{\mathbb{R}} e^{-C_1 |y|^2} dy \leq 0,$$

which implies the desired result.

4 Bismut derivative formula

In this section, we shall adopt the techniques of the Malliavin calculus to investigate the Bismut derivative formula and Harnack type inequality for P_T associated with (3.1). To this end, we make the following assumption: (H2)

- (i) there exist $x_0 \in \mathbb{R}$ and $p \geq 2$ such that $b(\cdot, x_0) \in L^p([0, T]; \mathbb{R})$;
- (ii) b is differentiable w.r.t. the space variable such that $\partial b(t, \cdot)$ is uniformly continuous uniformly w.r.t. the time variable t and moreover,

$$|\partial b(t, \cdot)(x)| \leq K_3, K_5 \leq |\sigma(t)^{-1}| \leq K_4, \quad \forall t \in [0, T], x \in \mathbb{R},$$

where K_3, K_4 and K_5 are positive constants.

Main result reads as follows.

Theorem 4.1 *Assume that (H2) holds. Then, for all $x, y \in \mathbb{R}$ and $f \in C_b^1(\mathbb{R})$,*

$$\nabla_y P_T f(x) = \mathbb{E} \left[f(X_T^x) \int_0^T \sigma(s)^{-1} \left(\left(1 + \int_0^s K_H(s, r) u'(r) dr \right) \partial b(s, \cdot)(X_s^x) - u'(s) \right) y dW_s \right],$$

where $u \in C^1([0, T]; \mathbb{R})$ such that $1 + \int_0^T K_H(T, r) u'(r) dr = 0$, X_s^x is the solution of (3.1).

The proof of this theorem is based in the following lemmas and proposition.

We first recall a result from [6, Theorem 4.1 and Theorem 4.2], in which the existence of Malliavin directional derivative is discussed.

Lemma 4.2 *Let b and σ be continuously differentiable w.r.t. their space variable, with bounded derivative; assume further that, there exist $x_0 \in \mathbb{R}$ and $p \geq 2$ such that $b(\cdot, x_0) \in L^p([0, T]; \mathbb{R})$ and σ is bounded. Then, for any $\xi \in \mathcal{H}$, $(\langle D_H X_t^x, \xi \rangle_{\mathcal{H}})_{t \in [0, T]}$ exists and is the unique solution to the equation*

$$Y_t = \langle K_H(K_H(t, \cdot)\sigma(\cdot, X_t^x)), \xi \rangle_{\mathcal{H}} + \int_0^t K_H(t, s) \partial b(s, \cdot)(X_s^x) Y_s ds + \int_0^t K_H(t, s) \partial \sigma(s, \cdot)(X_s^x) Y_s dW_s,$$

where X_t^x is the solution of (2.3).

Following the same method presented in [6, Theorem 3.3], we can show that the solution of (2.3) depends continuously on the initial condition in the sense specified below.

Lemma 4.3 *Assume b and σ are Lipschitz continuous for the second variable uniformly w.r.t. their first variable, and there exist $x_0, y_0 \in \mathbb{R}$ such that $b(\cdot, x_0) \in L_+^{\kappa_1}([0, T]; \mathbb{R})$, $\sigma(\cdot, y_0) \in L_+^{2\kappa_2}([0, T]; \mathbb{R})$. Denote by X^x and X^y the solution of (2.3) with initial condition x and y respectively. Then, for any $p \in A_H$, there exists constant $L_p > 0$ such that*

$$\sup_{t \in [0, T]} \mathbb{E} |X_t^x - X_t^y|^p \leq L_p |x - y|^p.$$

Remark 4.4 *If we consider the case $p = 2$, Lemma 4.3 reduces to [6, Theorem 3.3].*

Next we will concern the existence of the derivative process w.r.t. the initial data.

Proposition 4.5 *Suppose that b and σ are both differentiable w.r.t. their second variables such that $\partial b(t, \cdot)$ and $\partial \sigma(t, \cdot)$ are bounded and uniformly continuous uniformly w.r.t. their first variable t . Then, for each $y \in \mathbb{R}$, $(\nabla_y X_t^x)_{0 \leq t \leq T}$ exists and is the unique solution to the equation*

$$Y_t = y + \int_0^t K_H(t, s) \partial b(s, \cdot)(X_s^x) Y_s ds + \int_0^t K_H(t, s) \partial \sigma(s, \cdot)(X_s^x) Y_s dW_s,$$

where X_t^x is the solution of (2.3).

Proof. Using the Picard iteration argument introduced in [6, Theorem 3.1], we can easily show that the above equation has a unique solution $(Y_t)_{t \in [0, T]}$ and moreover, $\sup_{t \in [0, T]} \mathbb{E} |Y_t|^p < \infty$ holds for any $p \in A_H$.

For $\epsilon > 0$, let $Z_t^\epsilon = X_t^{x+\epsilon y} - X_t^x - \epsilon Y_t$, $t \in [0, T]$. To complete the proof, it suffices to prove

$$\lim_{\epsilon \rightarrow 0} \mathbb{E} \frac{|Z_t^\epsilon|^2}{\epsilon^2} = 0, \quad \forall t \in [0, T].$$

To this end, we see that, for any $t \in [0, T]$,

$$Z_t^\epsilon = \int_0^t K_H(t, s) (b(s, X_s^{x+\epsilon y}) - b(s, X_s^x) - \epsilon \partial b(s, \cdot)(X_s^x) Y_s) ds$$

$$+ \int_0^t K_H(t, s) (\sigma(s, X_s^{x+\epsilon y}) - \sigma(s, X_s^x) - \epsilon \partial \sigma(s, \cdot)(X_s^x) Y_s) dW_s.$$

Therefore, by the Hölder inequality and the Burkholder-Davis-Gundy inequality, there is some constant C_2 such that

$$\begin{aligned} \mathbb{E}|Z_t^\epsilon|^2 &\leq 2T \int_0^t K_H^2(t, s) |b(s, X_s^{x+\epsilon y}) - b(s, X_s^x) - \epsilon \partial b(s, \cdot)(X_s^x) Y_s|^2 ds \\ &\quad + 2C_2 \int_0^t K_H^2(t, s) |\sigma(s, X_s^{x+\epsilon y}) - \sigma(s, X_s^x) - \epsilon \partial \sigma(s, \cdot)(X_s^x) Y_s|^2 ds \\ &=: 2T \int_0^t K_H^2(t, s) J_1(s)^2 ds + 2C_2 \int_0^t K_H^2(t, s) J_2(s)^2 ds. \end{aligned} \quad (4.1)$$

Next we are to estimate $J_1(s)$ and $J_2(s)$. Let us define, for each $\delta \geq 0$,

$$\alpha(\delta) = \sup_{|x-y| \leq \delta} \sup_{s \in [0, T]} (|\partial b(s, \cdot)(x) - \partial b(s, \cdot)(y)| + |\partial \sigma(s, \cdot)(x) - \partial \sigma(s, \cdot)(y)|).$$

It is clear from the assumptions on the coefficients b and σ that $\alpha(\infty) < \infty$ and $\alpha(\delta) \downarrow 0$ as $\delta \downarrow 0$. As a consequence, we derive that,

$$\delta^2 \alpha^2(\delta) = \delta^2 \alpha^2(\delta) \mathbf{I}_{\{\delta \leq \sqrt{\epsilon}\}} + \delta^2 \alpha^2(\delta) \mathbf{I}_{\{\delta > \sqrt{\epsilon}\}} \leq \delta^2 \alpha^2(\sqrt{\epsilon}) + \frac{\delta^q \alpha^2(\infty)}{\epsilon^{\frac{q-2}{2}}},$$

where q is chosen such that $2 < q < \frac{1}{H_0}$.

Note that, by the mean value theorem, we get

$$J_1(s) = |(\partial b(s, \cdot)(\zeta_1) - \partial b(s, \cdot)(X_s^x)) (X_s^{x+\epsilon y} - X_s^x) + \partial b(s, \cdot)(X_s^x) Z_s^\epsilon|,$$

and

$$J_2(s) = |(\partial \sigma(s, \cdot)(\zeta_2) - \partial \sigma(s, \cdot)(X_s^x)) (X_s^{x+\epsilon y} - X_s^x) + \partial \sigma(s, \cdot)(X_s^x) Z_s^\epsilon|,$$

where $\zeta_i = \theta_i X_s^x + (1 - \theta_i) X_s^{x+\epsilon y}$, $\theta_i \in (0, 1)$, $i = 1, 2$.

Hence, we conclude that

$$\begin{aligned} J_1(s)^2 + J_2(s)^2 &\leq (J_1(s) + J_2(s))^2 \leq 2\alpha^2(|X_s^{x+\epsilon y} - X_s^x|) |X_s^{x+\epsilon y} - X_s^x|^2 + 2M |Z_s^\epsilon|^2 \\ &\leq 2\alpha^2(\sqrt{\epsilon}) |X_s^{x+\epsilon y} - X_s^x|^2 + \frac{\alpha^2(\infty)}{\epsilon^{\frac{q-2}{2}}} |X_s^{x+\epsilon y} - X_s^x|^q + 2M |Z_s^\epsilon|^2, \end{aligned} \quad (4.2)$$

where $M = \sup_{s \in [0, T]} (|\partial b(s, \cdot)| + |\partial \sigma(s, \cdot)|)^2$.

Now we turn to the estimate of $\mathbb{E}|Z_t^\epsilon|^2$. Substituting (4.2) into (4.1) and noting $q \in A_H$, we have, by Lemma 4.3,

$$\begin{aligned} \mathbb{E}|Z_t^\epsilon|^2 &\leq 2(T \vee C_2) T^{2H} \left(2L_2 |y|^2 \alpha^2(\sqrt{\epsilon}) \epsilon^2 + L_q |y|^q \alpha^2(\infty) \epsilon^{\frac{q}{2}+1} \right) + 4(T \vee C_2) M \int_0^t K_H^2(t, s) \mathbb{E}|Z_s^\epsilon|^2 ds \\ &=: C(\epsilon) + C_3 \int_0^t K_H^2(t, s) \mathbb{E}|Z_s^\epsilon|^2 ds. \end{aligned}$$

We set $K_1^2(t, s) = K_H^2(t, s)$, $K_{n+1}^2(t, s) = \int_s^t K_1^2(t, r) K_n^2(r, s) dr$, $\forall s, t \in [0, T]$, $n \geq 1$, and identify the operator K_n^2 with its kernel, $K_0^2 \equiv 1$.

Then by induction, we deduce that

$$\mathbb{E}|Z_t^\epsilon|^2 \leq C(\epsilon) (1 + C_3 (K_1^2 1)(t)) + C_3^2 \int_0^t \int_0^s K_H^2(t, s) K_H^2(s, r) \mathbb{E}|Z_r^\epsilon|^2 dr ds$$

$$\leq \cdots \leq C(\epsilon) \sum_{i=0}^n C_3^i (K_i^2 1)(t) + C_3^{n+1} \sup_{u \in [0, T]} \mathbb{E} |Z_u^\epsilon|^2 (K_{n+1}^2 1)(t).$$

Recall that [6, Lemma 3.3] states that $\sum_{i=0}^\infty \sup_{0 \leq t \leq T} (K_i^2 1)(t) z^i < \infty$, $\forall z \in \mathbb{C}$. Therefore, letting $n \rightarrow \infty$, it follows that

$$\mathbb{E} |Z_t^\epsilon|^2 \leq C(\epsilon) \sum_{i=0}^\infty C_3^i \sup_{t \in [0, T]} (K_i^2 1)(t).$$

Observing that $\lim_{\epsilon \rightarrow 0} \frac{C(\epsilon)}{\epsilon^2} = 0$, the proof is finished.

Now we are in position to prove Theorem 4.1.

Proof of Theorem 4.1. Note that if there exists $\xi \in \text{Dom} \delta_H$ such that

$$\langle D_H X_T^x, \xi \rangle_{\mathcal{H}} = \nabla_y X_T^x, \text{ a.s.}, \quad (4.3)$$

then for each $f \in C_b^1(\mathbb{R})$,

$$\begin{aligned} \nabla_y P_T f(x) &= \nabla_y \mathbb{E} f(X_T^x) = \mathbb{E} \nabla_y f(X_T^x) = \mathbb{E} (f'(X_T^x) \nabla_y X_T^x) \\ &= \mathbb{E} (f'(X_T^x) \langle D_H X_T^x, \xi \rangle_{\mathcal{H}}) = \mathbb{E} (\langle D_H f(X_T^x), \xi \rangle_{\mathcal{H}}). \end{aligned}$$

Applying the integration by parts formula for D_H , i.e. the definition of δ_H , we get

$$\nabla_y P_T f(x) = \mathbb{E} (f(X_T^x) \delta_H \xi) = \mathbb{E} \left(f(X_T^x) \int_0^T \dot{\xi}_s \delta_H B_s^H \right).$$

Furthermore, if $\dot{\xi} \in L_a^2([0, T] \times \Omega)$, then $\nabla_y P_T f(x) = \mathbb{E} \left(f(X_T^x) \int_0^T \dot{\xi}_s dW_s \right)$.

Based on the analysis above, we know that, to complete the proof, it suffices to find a $\xi = K_H \dot{\xi}$ such that $\dot{\xi} \in L_a^2([0, T] \times \Omega)$ and (4.3) holds.

Let

$$\dot{\xi}_s = \sigma(s)^{-1} \left(\left(1 + \int_0^s K_H(s, r) u'(r) dr \right) \partial b(s, \cdot)(X_s^x) - u'(s) \right) y,$$

where $u \in C^1([0, T]; \mathbb{R})$ such that $1 + \int_0^T K_H(T, r) u'(r) dr = 0$. Obviously, $\dot{\xi}$ constructed above is in $L_a^2([0, T] \times \Omega)$. Next consider the following equation

$$Z_t = y + \int_0^t K_H(t, s) \partial b(s, \cdot)(X_s^x) Z_s ds - \int_0^t K_H(t, s) \sigma(s) \dot{\xi}_s ds. \quad (4.4)$$

By the assumption, it is clear that (4.4) has a unique solution Z . On one hand, observe that $Y_t := (1 + \int_0^t K_H(t, r) u'(r) dr) y$ solves (4.4). On the other hand, since $\partial \sigma(s, \cdot) = 0$, $\forall s \in [0, T]$, Lemma 4.2 together with Proposition 4.5 implies that $(\nabla_y X_t^x - \langle D_H X_t^x, \xi \rangle_{\mathcal{H}})_{t \in [0, T]}$ is also a solution of (4.4). As a consequence, $Y_t = \nabla_y X_t^x - \langle D_H X_t^x, \xi \rangle_{\mathcal{H}}$, $\forall t \in [0, T]$ holds. Due to $Y_T = 0$, it follows that $\langle D_H X_T^x, \xi \rangle_{\mathcal{H}} = \nabla_y X_T^x$. Therefore, the proof is complete.

Remark 4.6 If we take $u'(t) = -\frac{C_H}{T} t^{\frac{1}{2}-H}$, then by the proof of Theorem 3.1, we know that $1 + \int_0^T K_H(T, r) u'(r) dr = 0$ holds and the result of Theorem 4.1 can be expressed as

$$\nabla_y P_T f(x) = \mathbb{E} \left[f(X_T^x) \int_0^T \sigma(s)^{-1} \left((T-s) \partial b(s, \cdot)(X_s^x) + C_H s^{\frac{1}{2}-H} \right) \frac{y}{T} dW_s \right].$$

In particular, when $H = \frac{1}{2}$, we obtain a version of relation above that is an extension of [38, Theorem 3.1], in which the coupling argument is used.

Next we will state some applications of the derivative formula obtained above. More precisely, explicit gradient estimate, Harnack inequality and log-Harnack inequality are presented. That is

Corollary 4.7 Assume that (H2) holds and set $C(T, K_3, K_4, H) = 2K_4^2 \left(\frac{K_3^2 T}{3} + \frac{C_H^2}{(2-2H)T^{2H}} \right)$.

(1) For any $f \in \mathcal{B}_b(\mathbb{R})$, we get

$$|\nabla_y P_T f(x)|^2 \leq C(T, K_3, K_4, H) |y|^2 P_T f^2(x),$$

i.e., $|\nabla_y P_T f(x)|$ is bounded above by f . Moreover, for all $\delta > 0$ and positive $f \in \mathcal{B}_b(\mathbb{R})$,

$$|\nabla_y P_T f(x)| \leq \delta [P_T(f \log f) - (P_T f)(\log P_T f)] + \frac{C(T, K_3, K_4, H)}{\delta} |y|^2 P_T f. \quad (4.5)$$

(2) For any non-negative $f \in \mathcal{B}_b(\mathbb{R})$ and $p > 1$, the following Harnack inequality holds:

$$(P_T f(x))^p \leq P_T f^p(y) \exp \left[\frac{p}{p-1} C(T, K_3, K_4, H) |x-y|^2 \right], \quad x, y \in \mathbb{R}. \quad (4.6)$$

As a consequence, the log-Harnack inequality

$$P_T(\log f)(x) \leq \log P_T f(y) + C(T, K_3, K_4, H) |x-y|^2 \quad (4.7)$$

holds for any positive $f \in \mathcal{B}_b(\mathbb{R})$, and P_T is strong Feller, i.e. for each $x \in \mathbb{R}$,

$$\lim_{y \rightarrow x} P_T f(y) = P_T f(x).$$

(3) Let μ be P_T sub-invariant, i.e., μ is a probability measure on \mathbb{R} such that $\int_{\mathbb{R}} P_T f d\mu \leq \int_{\mathbb{R}} f d\mu$ for all $f \in \mathcal{B}_b(\mathbb{R})$, $f \geq 0$. Then the entropy-cost inequality

$$\mu(P_T^* f \log P_T^* f) \leq C(T, K_3, K_4, H) W_2^d(f\mu, \mu)^2, \quad f \geq 0, \mu(f) = 1, \quad (4.8)$$

holds for the adjoint operator P_T^* of P_T in $L^2(\mu)$, where $d(x, y) = |x - y|$.

Proof. Let $u'(t) = -\frac{C_H}{T} t^{\frac{1}{2}-H}$ and define $M_T = \int_0^T \sigma(s)^{-1} \left((T-s) \partial b(s, \cdot)(X_s^x) + C_H s^{\frac{1}{2}-H} \right) \frac{y}{T} dW_s$. By the hypotheses on the coefficients, we derive that

$$\langle M \rangle_T = \int_0^T \left| \sigma(s)^{-1} \left((T-s) \partial b(s, \cdot)(X_s^x) + C_H s^{\frac{1}{2}-H} \right) \frac{y}{T} \right|^2 ds \leq C(T, K_3, K_4, H) |y|^2,$$

where $C(T, K_3, K_4, H) = 2K_4^2 \left(\frac{K_3^2 T}{3} + \frac{C_H^2}{(2-2H)T^{2H}} \right)$.

Hence, it follows from the Hölder inequality that

$$|\nabla_y P_T f(x)|^2 \leq \mathbb{E} \langle M \rangle_T P_T f^2(x) \leq C(T, K_3, K_4, H) |y|^2 P_T f^2(x).$$

Combining the derivative formula with the Young inequality yield that, for any positive $f \in \mathcal{B}_b(\mathbb{R})$ and $\delta > 0$,

$$|\nabla_y P_T f(x)| \leq \delta [P_T(f \log f) - (P_T f)(\log P_T f)] + \delta \log \mathbb{E} \exp \left[\frac{M_T}{\delta} \right] P_T f. \quad (4.9)$$

Observe that

$$\mathbb{E} \exp \left[\frac{M_T}{\delta} \right] \leq \left(\mathbb{E} \exp \left[\frac{2\langle M \rangle_T}{\delta^2} \right] \right)^{\frac{1}{2}} \leq \exp \left[\frac{C(T, K_3, K_4, H)}{\delta^2} |y|^2 \right]. \quad (4.10)$$

Substituting (4.10) into (4.9) implies (4.5). In the spirit of [40, Corollary 1.2], (4.6) follows from (4.5). Since \mathbb{R} is a length space, then according to [37, Proposition 2.2], (4.6) implies (4.7). The strong Feller property follows from (4.6), due to the same proof of [8, Proposition 4.1]. Finally, (4.8) can be proved as the proof of [28, Corollary 1.2] or [14, Corollary 3.6].

Remark 4.8 *Making use of the Harnack inequality, one can compare the values of a reference function at different points, while in the shift Harnack inequality presented in Theorem 3.1, instead of initial points, a reference function is shifted. Besides, the (resp. shift) Harnack inequality allows us to compare the measure $P_T(x, \cdot)$ with some invariant probability measure associated with a certain semigroup (resp. the Lebesgue measure), where $P_T(x, \cdot)$ is the transition probability for P_T . One can see [39] for more applications of the shift Harnack inequality.*

5 Transportation inequalities

In this section we will discuss the Talagrand type transportation cost inequalities for the law of the solution of (2.3) w.r.t. the uniform distance d_∞ and the L^2 -distance d_2 on the path space $C([0, T]; \mathbb{R})$. To the end, we introduce the following assumption: (H3)

(i) there exists constant $K_6(> 0)$ such that

$$|b(t, x) - b(t, y)| + |\sigma(t, x) - \sigma(t, y)| \leq K_6 |x - y|, \quad \forall t \in [0, T], \quad x, y \in \mathbb{R};$$

(ii) $\|\sigma\|_\infty := \sup_{0 \leq t \leq T} \sup_{x \in \mathbb{R}} |\sigma(t, x)| < \infty$.

Let us start by prove the following proposition which is crucial for the proof of Theorem 5.2 below.

Proposition 5.1 *Let $H > \frac{1}{2}$ and τ be an (\mathcal{F}_t) -stopping time. Assume that ϕ is an adapted stochastic process satisfying $\mathbb{E} \int_0^T |\phi_t|^p dt < \infty$ for some $p \geq 2$. Then, there holds the maximal inequality*

$$\mathbb{E} \left(\sup_{0 \leq t \leq T \wedge \tau} \left| \int_0^t K_H(t, s) \phi(s) dW_s \right|^p \right) \leq C(p) \mathbb{E} \int_0^{T \wedge \tau} |\phi_t|^p dt,$$

where $C(p)$ is a positive constant depending on p .

Proof. Recall that, for $H \in (0, 1)$, $K_H(t, s)$ is the kernel

$$K_H(t, s) = \alpha_H (t - s)^{H - \frac{1}{2}} + \alpha_H \left(\frac{1}{2} - H \right) \int_s^t (r - s)^{H - \frac{3}{2}} \left(1 - \left(\frac{s}{r} \right)^{\frac{1}{2} - H} \right) dr,$$

where $\alpha_H = \left(\frac{2H\Gamma(3/2-H)}{\Gamma(H+1/2)\Gamma(2-2H)} \right)^{1/2}$.

From this relation, we get

$$\frac{\partial K_H(t, s)}{\partial t} = \alpha_H \left(H - \frac{1}{2} \right) \left(\frac{s}{t} \right)^{\frac{1}{2} - H} (t - s)^{H - \frac{3}{2}}.$$

When $H > \frac{1}{2}$, the kernel $K_H(t, s)$ can reformulate as (for instance, see [1] and references therein)

$$K_H(t, s) = \alpha_H \left(H - \frac{1}{2} \right) s^{\frac{1}{2}-H} \int_s^t r^{H-\frac{1}{2}} (r-s)^{H-\frac{3}{2}} dr =: \bar{\alpha}_H s^{\frac{1}{2}-H} \int_s^t r^{H-\frac{1}{2}} (r-s)^{H-\frac{3}{2}} dr.$$

Therefore, we have

$$\int_0^t K_H(t, s) \phi(s) dW_s = \bar{\alpha}_H \int_0^t s^{\frac{1}{2}-H} \int_s^t r^{H-\frac{1}{2}} (r-s)^{H-\frac{3}{2}} dr \phi(s) dW_s.$$

To exchange the integration of the right-hand side of the above expression, one need to show that the integrand fulfills the conditions of the stochastic Fubini theorem (see [9, Theorem 4.18]). Actually, choosing $\epsilon \in (0, \frac{1}{2})$ such that $H > \frac{1+\epsilon}{2}$ and using the Hölder inequality and the Young inequality, we obtain

$$\begin{aligned} & \int_0^t r^{H-\frac{1}{2}} \left(\mathbb{E} \int_0^r s^{1-2H} (r-s)^{2H-3} \phi^2(s) ds \right)^{\frac{1}{2}} dr \\ & \leq \left(\int_0^t r^{-1+2\epsilon} dr \right)^{\frac{1}{2}} \left(\int_0^t r^{2(H-\epsilon)} \mathbb{E} \int_0^r s^{1-2H} (r-s)^{2H-3} \phi^2(s) ds dr \right)^{\frac{1}{2}} \\ & \leq \left(\frac{T^{2\epsilon}}{2\epsilon} \right)^{\frac{1}{2}} \left(\int_0^t r^{4H-2\epsilon-3} dr \mathbb{E} \int_0^t r^{1-2\epsilon} \phi^2(r) dr \right)^{\frac{1}{2}} \\ & \leq \left(\frac{T^{4H-2\epsilon-1}}{4\epsilon(2H-\epsilon-1)} \right)^{\frac{1}{2}} \left(\mathbb{E} \int_0^T \phi^2(r) dr \right)^{\frac{1}{2}}, \end{aligned}$$

which is finite due to hypothesis on ϕ .

So, the stochastic Fubini theorem implies

$$\int_0^t K_H(t, s) \phi(s) dW_s = \bar{\alpha}_H \int_0^t r^{H-\frac{1}{2}} \int_0^r s^{\frac{1}{2}-H} (r-s)^{H-\frac{3}{2}} \phi(s) dW_s dr.$$

Taking $\theta \in (0, \frac{1}{2})$ such that $H > \frac{1+\theta}{2}$ and applying the Hölder inequality and the Young inequality, we obtain

$$\begin{aligned} & \mathbb{E} \left(\sup_{0 \leq t \leq T \wedge \tau} \left| \int_0^t r^{H-\frac{1}{2}} \int_0^r s^{\frac{1}{2}-H} (r-s)^{H-\frac{3}{2}} \phi(s) dW_s dr \right|^p \right) \\ & \leq \left(\int_0^T r^{(\frac{1}{p}-1+\theta)\frac{p}{p-1}} dr \right)^{p-1} \int_0^T r^{(H+\frac{1}{2}-\frac{1}{p}-\theta)p} \mathbb{E} \left(\left| \int_0^r s^{\frac{1}{2}-H} (r-s)^{H-\frac{3}{2}} \phi(s) dW_s \right|^p \cdot \mathbf{I}_{[0, T \wedge \tau]}(r) \right) dr \\ & \leq \left(\int_0^T r^{(\frac{1}{p}-1+\theta)\frac{p}{p-1}} dr \right)^{p-1} \int_0^T r^{(H+\frac{1}{2}-\frac{1}{p}-\theta)p} \mathbb{E} \left| \int_0^{r \wedge \tau} s^{\frac{1}{2}-H} (r-s)^{H-\frac{3}{2}} \phi(s) dW_s \right|^p dr \\ & \leq \left(\frac{p-1}{\theta p} \right)^{p-1} T^{\theta p} \int_0^T r^{(H+\frac{1}{2}-\frac{1}{p}-\theta)p} \mathbb{E} \left(\int_0^{r \wedge \tau} s^{1-2H} (r-s)^{2H-3} \phi(s)^2 ds \right)^{\frac{p}{2}} dr \\ & = \left(\frac{p-1}{\theta p} \right)^{p-1} T^{\theta p} \int_0^T r^{(H+\frac{1}{2}-\frac{1}{p}-\theta)p} \mathbb{E} \left(\int_0^T s^{1-2H} (r-s)^{2H-3} \phi(s)^2 \mathbf{I}_{\{s \leq r\}} \mathbf{I}_{\{s \leq T \wedge \tau\}} ds \right)^{\frac{p}{2}} dr \\ & \leq \left(\frac{p-1}{\theta p} \right)^{p-1} T^{\theta p} \left(\int_0^T r^{(H+\frac{1}{2}-\frac{1}{p}-\theta)p+2H-3} dr \right)^{\frac{p}{2}} \mathbb{E} \int_0^T r^{(H+\frac{1}{2}-\frac{1}{p}-\theta)p+(1-2H)\frac{p}{2}} |\phi_r|^p \mathbf{I}_{\{r \leq T \wedge \tau\}} dr \\ & \leq \left(\frac{p-1}{\theta p} \right)^{p-1} \left(\frac{1}{(H+\frac{1}{2}-\theta)p+2H-3} \right)^{\frac{p}{2}} T^{(H+\frac{1}{2}-\theta)\frac{p}{2}+(H-\frac{1}{2})p-1} \mathbb{E} \int_0^{T \wedge \tau} |\phi_r|^p dr, \end{aligned}$$

which yields the desired result.

We now prove the following main result in this section.

Theorem 5.2 Let $H > \frac{1}{2}$. Assume (H3) and let \mathbb{P}_x be the law of the solution of (2.3) with the initial point x on the path space $C([0, T]; \mathbb{R})$. Then, \mathbb{P}_x satisfies the transportation cost inequalities on the metric space $C([0, T]; \mathbb{R})$. More precisely,

- (1) $\mathbb{P}_x \in T_2(\alpha(T, H)|d_\infty)$, where $\alpha(T, H) = 3(\|\sigma\|_\infty T^H)^2 e^{3K_6^2 T(T^{2H} + C(2))}$,
- (2) $\mathbb{P}_x \in T_2(\beta(T, H)|d_2)$, where $\beta(T, H) = 3(\|\sigma\|_\infty T^H)^2 \frac{e^{3K_6^2 T(T^{2H} + C(2))} - 1}{3K_6^2(T^{2H} + C(2))}$.

Proof. Let \mathbb{Q} be a probability measure on $C([0, T]; \mathbb{R})$ such that $\mathbb{Q} \ll \mathbb{P}_x$. Clearly, to prove the desired result, we only need to consider the case $\mathbb{H}(\mathbb{Q}|\mathbb{P}_x) < \infty$. The proof will divide into two steps.

Step 1. The part follows the arguments of [11]. Let $\bar{\mathbb{Q}} = \frac{d\mathbb{Q}}{d\mathbb{P}_x}(X.)\mathbb{P}$. Note that

$$\int_{\Omega} \frac{d\mathbb{Q}}{d\mathbb{P}_x}(X.)d\mathbb{P} = \int_{C([0, T]; \mathbb{R})} \frac{d\mathbb{Q}}{d\mathbb{P}_x}(\gamma)d\mathbb{P}_x(\gamma) = \mathbb{Q}(C([0, T]; \mathbb{R})) = 1,$$

and

$$\int_{C([0, T]; \mathbb{R})} \frac{d\mathbb{Q}}{d\mathbb{P}_x} \log \left(\frac{d\mathbb{Q}}{d\mathbb{P}_x} \right) d\mathbb{P}_x = \int_{\Omega} \frac{d\mathbb{Q}}{d\mathbb{P}_x}(X^x) \log \left(\frac{d\mathbb{Q}}{d\mathbb{P}_x}(X^x) \right) d\mathbb{P} = \int_{\Omega} \frac{d\bar{\mathbb{Q}}}{d\mathbb{P}} \log \left(\frac{d\bar{\mathbb{Q}}}{d\mathbb{P}} \right) d\mathbb{P}$$

that is, $\bar{\mathbb{Q}}$ is a probability measure on (Ω, \mathcal{F}) and $\mathbb{H}(\mathbb{Q}|\mathbb{P}_x) = \mathbb{H}(\bar{\mathbb{Q}}|\mathbb{P})$.

According to the proof of [11, Theorem 5.6], there is a predictable process $(u_t)_{0 \leq t \leq T}$ such that

$$\mathbb{H}(\mathbb{Q}|\mathbb{P}_x) = \mathbb{H}(\bar{\mathbb{Q}}|\mathbb{P}) = \frac{1}{2} \mathbb{E}_{\bar{\mathbb{Q}}} \int_0^T |u_t|^2 dt$$

and the process

$$\bar{W}_t := W_t - \int_0^t u_s ds$$

is a Brownian motion under $\bar{\mathbb{Q}}$, where $\mathbb{E}_{\bar{\mathbb{Q}}}$ is the expectation taken for the probability measure $\bar{\mathbb{Q}}$. As a consequence, the process $(\bar{B}_t^H)_{0 \leq t \leq T}$ defined by

$$\bar{B}_t^H = \int_0^t K_H(t, s) d\bar{W}_s = B_t^H - (K_H u)(t)$$

is a $\bar{\mathbb{Q}}$ -fractional Brownian motion associated with \bar{W} .

Step 2. From step 1, we can reformulate (2.3) as

$$X_t = x + \int_0^t K_H(t, s) (b(s, X_s) + \sigma(s, X_s)u_s) ds + \int_0^t K_H(t, s) \sigma(s, X_s) d\bar{W}_s. \quad (5.1)$$

Noting that, for any bounded measurable function F on $C([0, T]; \mathbb{R})$,

$$\mathbb{E}_{\bar{\mathbb{Q}}}(F(X.)) = \mathbb{E} \left(\frac{d\mathbb{Q}}{d\mathbb{P}_x}(X.) F(X.) \right) = \int_{C([0, T]; \mathbb{R})} \frac{d\mathbb{Q}}{d\mathbb{P}_x}(\gamma) F(\gamma) d\mathbb{P}_x(\gamma) = \mathbb{Q}(F),$$

it follows that the law of X under $\bar{\mathbb{Q}}$ is \mathbb{Q} . On the other hand, we consider the following equation

$$Y_t = x + \int_0^t K_H(t, s) b(s, Y_s) ds + \int_0^t K_H(t, s) \sigma(s, Y_s) d\bar{W}_s. \quad (5.2)$$

As \bar{W} is the Brownian motion under $\bar{\mathbb{Q}}$, we easily know that the law of Y under $\bar{\mathbb{Q}}$ is \mathbb{P}_x . Therefore, the law of (X, Y) under $\bar{\mathbb{Q}}$ is a coupling of $(\mathbb{Q}, \mathbb{P}_x)$ and moreover, we get

$$\begin{aligned} W_2^{d_\infty}(\mathbb{Q}, \mathbb{P}_x)^2 &\leq \mathbb{E}_{\bar{\mathbb{Q}}} d_\infty(X, Y)^2 = \mathbb{E}_{\bar{\mathbb{Q}}} \left(\sup_{0 \leq t \leq T} |X_t - Y_t|^2 \right), \\ W_2^{d_2}(\mathbb{Q}, \mathbb{P}_x)^2 &\leq \mathbb{E}_{\bar{\mathbb{Q}}} d_2(X, Y)^2 = \mathbb{E}_{\bar{\mathbb{Q}}} \left(\int_0^T |X_t - Y_t|^2 dt \right). \end{aligned}$$

Combining (5.1) with (5.2), we have

$$\begin{aligned} X_t - Y_t &= \int_0^t K_H(t, s) (b(s, X_s) - b(s, Y_s)) ds + \int_0^t K_H(t, s) \sigma(s, X_s) u_s ds \\ &\quad + \int_0^t K_H(t, s) (\sigma(s, X_s) - \sigma(s, Y_s)) d\bar{W}_s. \end{aligned}$$

Now, for $n \in \mathbb{N}$, define the stopping time

$$\tau_n := \inf\{t > 0, |X_t - Y_t| \geq n\}.$$

Obviously, $\tau_n \uparrow \infty$ as n goes to ∞ . Applying Proposition 5.1 and the Hölder inequality, we derive that

$$\begin{aligned} \mathbb{E}_{\bar{\mathbb{Q}}} \left(\sup_{0 \leq t \leq T} |(X_t - Y_t) \mathbf{I}_{\{t \leq \tau_n\}}|^2 \right) &\leq 3 \mathbb{E}_{\bar{\mathbb{Q}}} \left(\sup_{0 \leq t \leq T} \int_0^{t \wedge \tau_n} |K_H(t, s) (b(s, X_s) - b(s, Y_s))| ds \right)^2 \\ &\quad + 3 \mathbb{E}_{\bar{\mathbb{Q}}} \left(\sup_{0 \leq t \leq T} \int_0^{t \wedge \tau_n} |K_H(t, s) \sigma(s, X_s) u_s| ds \right)^2 \\ &\quad + 3 \mathbb{E}_{\bar{\mathbb{Q}}} \left(\sup_{0 \leq t \leq T} \left| \mathbf{I}_{\{t \leq \tau_n\}} \cdot \int_0^t K_H(t, s) (\sigma(s, X_s) - \sigma(s, Y_s)) d\bar{W}_s \right|^2 \right) \\ &\leq 3K_6^2 \mathbb{E}_{\bar{\mathbb{Q}}} \left(\sup_{0 \leq t \leq T} \int_0^{t \wedge \tau_n} |K_H(t, s)| \cdot |X_s - Y_s| ds \right)^2 \\ &\quad + 3\|\sigma\|_\infty^2 \mathbb{E}_{\bar{\mathbb{Q}}} \left(\sup_{0 \leq t \leq T} \int_0^{t \wedge \tau_n} |K_H(t, s) u_s| ds \right)^2 \\ &\quad + 3C(2) \mathbb{E}_{\bar{\mathbb{Q}}} \left(\int_0^{T \wedge \tau_n} |\sigma(s, X_s) - \sigma(s, Y_s)|^2 ds \right) \\ &\leq 3(\|\sigma\|_\infty T^H)^2 \mathbb{E}_{\bar{\mathbb{Q}}} \int_0^T u_s^2 ds \\ &\quad + 3K_6^2 (T^{2H} + C(2)) \mathbb{E}_{\bar{\mathbb{Q}}} \left(\int_0^{T \wedge \tau_n} |X_s - Y_s|^2 ds \right) \\ &= 3(\|\sigma\|_\infty T^H)^2 \mathbb{E}_{\bar{\mathbb{Q}}} \int_0^T u_s^2 ds \\ &\quad + 3K_6^2 (T^{2H} + C(2)) \mathbb{E}_{\bar{\mathbb{Q}}} \left(\int_0^T |(X_s - Y_s) \mathbf{I}_{\{s \leq \tau_n\}}|^2 ds \right) \\ &\leq 3(\|\sigma\|_\infty T^H)^2 \mathbb{E}_{\bar{\mathbb{Q}}} \int_0^T u_s^2 ds \\ &\quad + 3K_6^2 (T^{2H} + C(2)) \int_0^T \mathbb{E}_{\bar{\mathbb{Q}}} \left(\sup_{0 \leq t \leq s} |(X_t - Y_t) \mathbf{I}_{\{t \leq \tau_n\}}|^2 \right) ds. \end{aligned}$$

By the Gronwall inequality, we obtain

$$\mathbb{E}_{\bar{\mathbb{Q}}} \left(\sup_{0 \leq t \leq T} |(X_t - Y_t) \mathbf{I}_{\{t \leq \tau_n\}}|^2 \right) \leq 3(\|\sigma\|_{\infty} T^H)^2 \exp[3K_6^2 T(T^{2H} + C(2))] \mathbb{E}_{\bar{\mathbb{Q}}} \int_0^T u_s^2 ds.$$

The Fatou lemma leads to

$$\mathbb{E}_{\bar{\mathbb{Q}}} \left(\sup_{0 \leq t \leq T} |X_t - Y_t|^2 \right) \leq 3(\|\sigma\|_{\infty} T^H)^2 \exp[3K_6^2 T(T^{2H} + C(2))] \mathbb{E}_{\bar{\mathbb{Q}}} \int_0^T u_s^2 ds.$$

Hence, we deduce that

$$W_2^{d_{\infty}}(\mathbb{Q}, \mathbb{P}_x)^2 \leq 2\alpha(T, H) \mathbb{H}(\mathbb{Q} | \mathbb{P}_x)$$

with $\alpha(T, H) = 3(\|\sigma\|_{\infty} T^H)^2 \exp[3K_6^2 T(T^{2H} + C(2))]$.

For the metric d_2 , using the above procedure, we also can prove

$$\begin{aligned} \mathbb{E}_{\bar{\mathbb{Q}}} \left(|(X_t - Y_t) \mathbf{I}_{\{t \leq \tau_n\}}|^2 \right) &\leq 3(\|\sigma\|_{\infty} T^H)^2 \int_0^t \mathbb{E}_{\bar{\mathbb{Q}}} u_s^2 ds \\ &\quad + 3K_6^2 (T^{2H} + C(2)) \int_0^t \mathbb{E}_{\bar{\mathbb{Q}}} \left(|(X_s - Y_s) \mathbf{I}_{\{s \leq \tau_n\}}|^2 \right) ds. \end{aligned}$$

The Gronwall inequality, together with the Fatou lemma, yields

$$\mathbb{E}_{\bar{\mathbb{Q}}} (|X_t - Y_t|^2) \leq 3(\|\sigma\|_{\infty} T^H)^2 \int_0^t \exp[3K_6^2 (T^{2H} + C(2))(t-s)] \mathbb{E}_{\bar{\mathbb{Q}}} u_s^2 ds.$$

Thus it follows that

$$\begin{aligned} W_2^{d_2}(\mathbb{Q}, \mathbb{P}_x)^2 &\leq \mathbb{E}_{\bar{\mathbb{Q}}} \left(\int_0^T |X_t - Y_t|^2 dt \right) \\ &\leq 3(\|\sigma\|_{\infty} T^H)^2 \mathbb{E}_{\bar{\mathbb{Q}}} \int_0^T u_s^2 \left(\int_s^T \exp[3K_6^2 (T^{2H} + C(2))(t-s)] dt \right) ds \\ &\leq 2\beta(T, H) \mathbb{H}(\mathbb{Q} | \mathbb{P}_x), \end{aligned}$$

where $\beta(T, H) = 3(\|\sigma\|_{\infty} T^H)^2 \frac{e^{3K_6^2 T(T^{2H} + C(2))} - 1}{3K_6^2 (T^{2H} + C(2))}$. The proof is complete.

Remark 5.3 In general, $\int_0^{t \wedge \tau_n} K_H(t \wedge \tau_n, s) f(X_s, Y_s) d\bar{W}_s$ does not make sense, which forces us to consider $\mathbb{E}_{\bar{\mathbb{Q}}} \left(|(X_t - Y_t) \mathbf{I}_{\{t \leq \tau_n\}}|^2 \right)$ rather than $\mathbb{E}_{\bar{\mathbb{Q}}} (|X_{t \wedge \tau_n} - Y_{t \wedge \tau_n}|^2)$. Further reading on stochastic Volterra equation, one can see [44] and references therein.

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